HORNSBY GIRLS HIGH SCHOOL

Mathematics Advanced

Year 12 Higher School Certificate
Trial Examination Term 3 2023

STUDENT NUMBER: ____________________

General Instructions:

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks (pages 3–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–30)

- Attempt Questions 11–33
- Allow about 2 hour and 45 minutes for this section

Q1 - 10	Q11 - 13	Q14 - 18	Q19 - 22	Q23 - 27	Q28	Q29 - 31	Q32 – 33	Total
/10	/9	/15	/14	/23	/7	/12	/10	/100

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. Let $f(x)=\frac{1}{\sqrt{x}}, g(x)=1-x$, the range of $g(f(x))$ is:

   A. $x>0$

   B. $y\neq 0$

   C. $y\in\mathbb{R}$

   D. $y<1$

2. The following table represents a probability distribution with a mean of 1.2.

x	*0*	*1*	*2*
$P(X=x)$	*a*	*b*	*0.3*

   The value of a and b are:

   A. $a=0.1,\ \ b=0.6$

   B. $a=0.6,\ \ b=0.1$

   C. $a=0.3,\ \ b=0.4$

   D. $a=0.5,\ \ b=0.7$

3. It is known that for a particular function, $y=f(x)$, that

$$f(2)=-5$$
$$f'(2)=0$$
$$f''(2)=3$$

   Which statement below is true regarding the graph of $y=f(x)$?

   A. It passes through the point $(2,0)$.

   B. There is a local minimum at $(2,-5)$.

   C. It's concave down at $(2,-5)$.

   D. There is a point of inflexion at $(2,4)$.

4. Which one of the following is equivalent to $\cot\theta$?

A. $\tan(2\pi-\theta)$

B. $\dfrac{1}{\tan\left(\dfrac{\pi}{2}-\theta\right)}$

C. $\sec^2\theta-1$

D. $\sin\theta\cos\theta\ \text{cosec}^2\theta$

5. The graph of the function $y=f(x)$ is known to have a minimum turning pointing at the point $P(-4,-8)$. Therefore, the graph of $y=-f(2x)$ will have a maximum turning point at:

A. $(-2,8)$

B. $(-8,8)$

C. $(-4,8)$

D. $(2,-8)$

6. The solution of $|x-1|>2$ is:

A. $x>3$

B. $x<-1, x>3$

C. $x<-1$

D. $x\le -1, x\ge 3$

7. Which one of the following statements is not true?

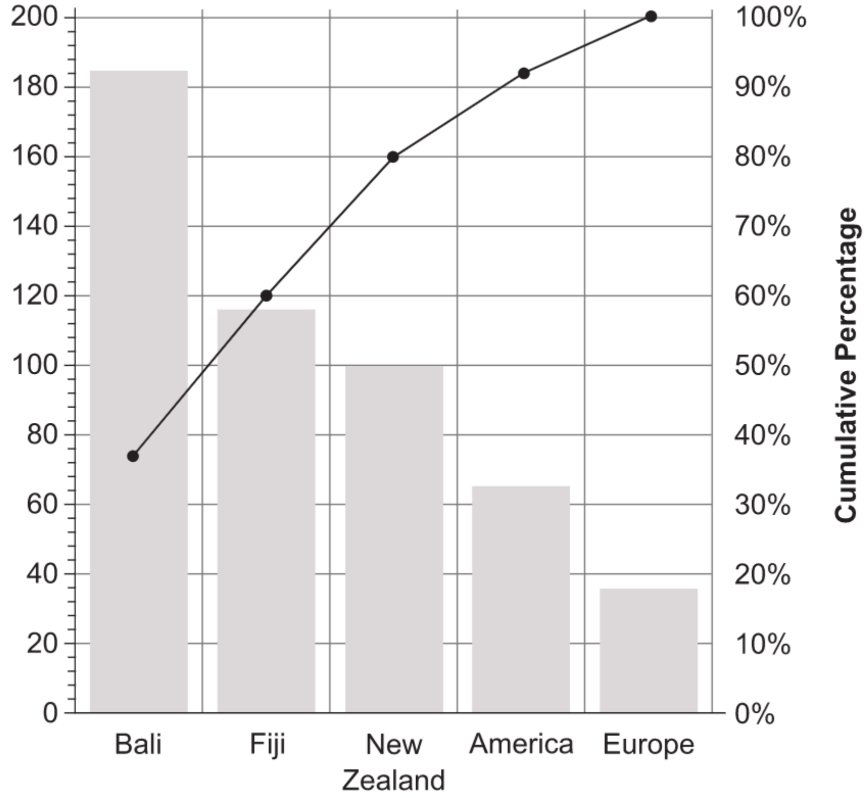
A. $\int_{-3}^{3} x^3 dx = 0$

B. $\int_{-3}^{3} x^2 dx = 2\int_{-3}^{0} x^2 dx$

C. $\int_{0}^{3} x dx = -\int_{-3}^{0} x dx$

D. $\int_{-1}^{2} 3 dx = 6$

8. The Pareto chart below shows the data collected from a survey where people were asked to choose their favourite overseas holiday destination.



Using the chart, which one of the following is true?

A. 80% of the people chose Bali and Fiji as their favourite holiday destination.

B. 50% of the people chose New Zealand as their favourite holiday destination.

C. 100 people chose Fiji as their favourite holiday destination.

D. A total of 500 people were surveyed.

9. For which domain is the expression $\cos\theta + \cot\theta < 0$ true?

A. $\left(0, \frac{\pi}{2}\right)$

B. $\left(\pi, \frac{3\pi}{2}\right)$

C. $\left(\frac{\pi}{2}, \pi\right)$

D. $\left(-\pi, -\frac{\pi}{2}\right)$

10. Which function is a primitive of $4x\left(x^2+3\right)^5$?

A. $12\left(x^2+1\right)\left(x^2+3\right)^4 + C$

B. $\frac{\left(x^2+3\right)^6}{3} + C$

C. $\frac{2\left(x^2+3\right)^6}{3} + C$

D. $\left(x^2+3\right)^6 + C$

End of Section I

Section II 90 marks

Attempt all questions. Allow about 2 hours and 45 minutes for this section

Answer each question in the spaces provided.

Your responses should include relevant mathematical reasoning and/or calculations.

Extra writing space is provided at the back of the examination paper.

Question 11 (3 Marks)

Twenty people had their heart rate measured in beats per minute after a period of exercise.

The information is displayed below in a stem and leaf plot.

Heart Rate After Exercise

8	**6 6 7 8 9**
9	**0 3 4 4 5 8 8 9**
10	**1 3 3**
11	**7**
12	**4 5**
13	
14	**3**

(a) Use calculations to explain why the heart rate of 143 beats per minute after exercise could be considered an outlier for the recorded data. **2**

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(b) Describe the skewness of the heart rate data after exercise. **1**

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Question 12 (4 Marks)

Metal spherical balls are manufactured so that their weight, W, varies directly with the cube of their radii, r. A ball manufactured with a radius of 2.5 cm, has a weight of 1.25 kg. **2**

(a) Calculate the radius of a ball manufactured with a weight of 5.12 kg.

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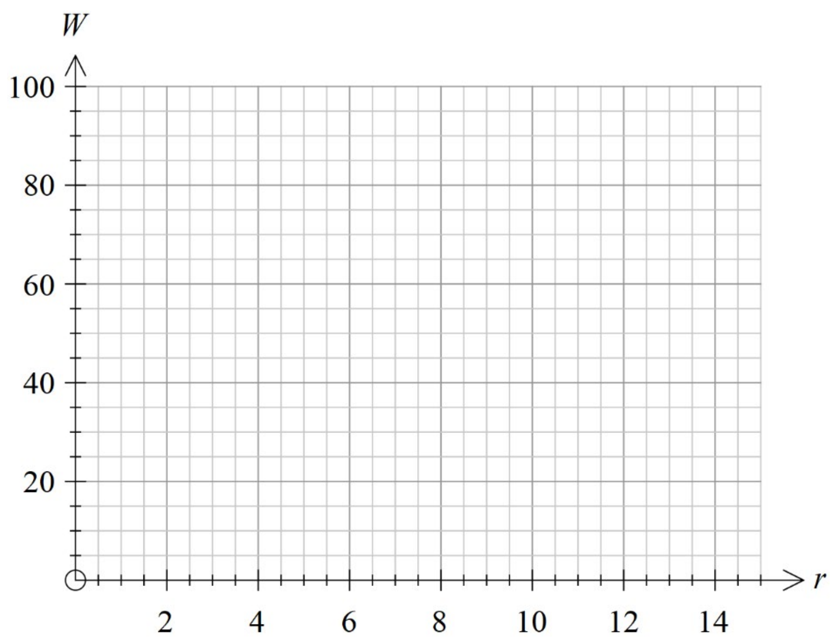
(b) Graph the relationship between the radius and weight of the metal spherical balls from $r = 0$ cm to $r = 10$ cm. **2**



Question 13 (2 Marks)

Use the trapezoidal rule with three subintervals to find an approximate value of **2**

$$\int_1^7 \frac{1}{\sqrt{2x-1}}\,dx.$$

Give your answer correct to 2 decimal places.

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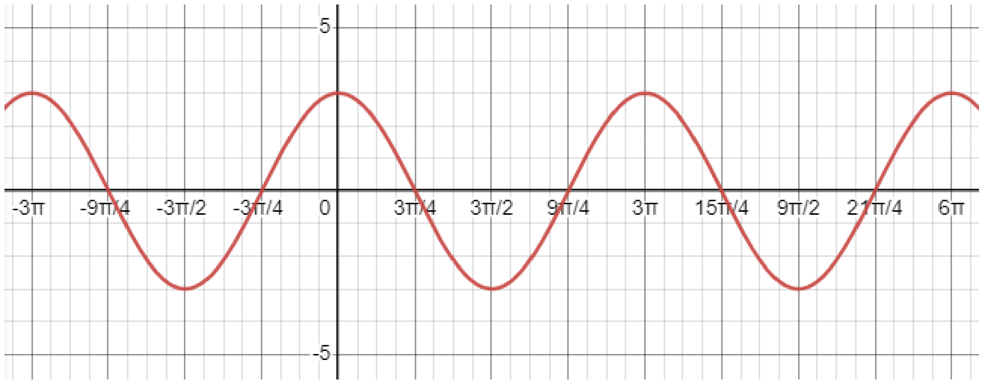
Question 14 (2 Marks)

The graph of $y = k\cos nx$ is shown. **2**



NOT TO SCALE

What are the values of k and n?

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Question 15 (3 Marks)

Three standard six-sided dice are thrown once.

a) What is the probability that all three dice show six? **1**

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b) What is the probability that exactly two of the dice show six? **1**

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c) What is the probability of rolling three different numbers? **1**

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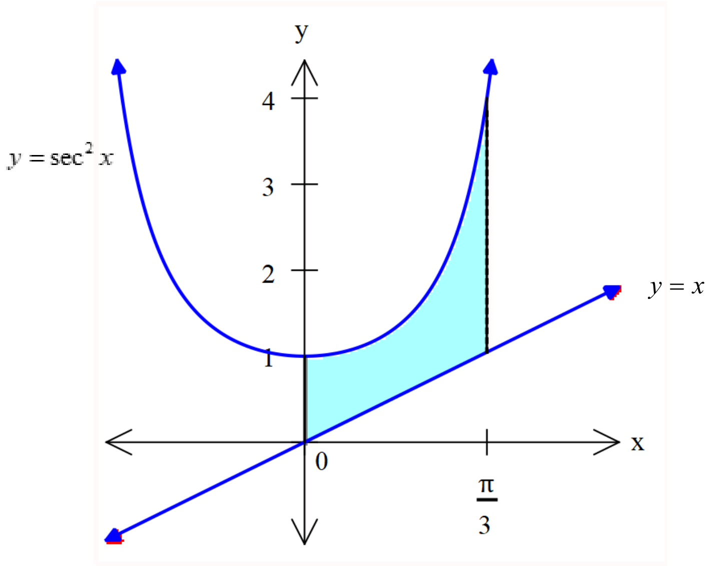
Question 16 (3 Marks)

The diagram shows the graphs of the functions $y = \sec^2 x$ and $y = x$. **3**

Calculate the exact area of the shaded region between $x = 0$ and $x = \frac{\pi}{3}$.



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Question 17 (4 Marks)

AB is a vertical tower 30 m high, C and D are points at the same level as the foot of tower, A. The point C is due south of the tower and D is due west of C. If the angles of elevation of C and D to the top of the tower are 16^0 and 11^0 respectively, find the distance CD and the bearing of D from the tower. Correct your answer to 2 decimal places and nearest minute respectively. **4**

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Question 18 (3 Marks)

(a) Differentiate $y = x\tan x$ **1**

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(b) Hence, or otherwise, find $\int x\sec^2 x dx$. **2**

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Question 19 (3 Marks)

The curve $y = kx^2 + c$ is subject to the following transformations: **3**

- Translated 2 units to the right.
- Dilated vertically by a factor of 4.
- Reflected in the ***y***-axis.

The final equation of the curve is $y = 8x^2 + 32x - 8$.

Find the values of ***k*** and ***c***.

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Question 20 (4 Marks)

The population of magpies, P, in a suburb is modelled by the function

$P = \dfrac{300}{1+14e^{-0.5t}}$, where t is the time in years.

(a) Find the range of the function P. **2**

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(b) What is the rate of change of the population of magpies 5 years later? **2**
Answer correct to the nearest whole number.

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Question 21 (4 Marks)

(a) A university is made up of local and international students. 85% are local students of whom 25% hold a scholarship. Of the international students, 45% are scholarship holders. If a randomly chosen student holds a scholarship, what is the probability that he or she is an international student? **2**

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(b) Suppose that A and B are not independent events. **2**

$P(A)=\frac{1}{4}$, $P(B)=\frac{2}{3}$, $P(A\cup B)=\frac{3}{4}$.

If $\overline{A}$ denotes the complement of A, find $P\left(\overline{A}\cap B\right)$.

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Question 22 (3 Marks)

Find the global maximum and minimum value of **3**

$$y = \begin{cases} -2, & \text{for } x < -3 \\ x+1, & \text{for } -3 \le x < 1 \\ -x^3 - x^2 + 4, & \text{for } 1 \le x \le 1.5 \end{cases}$$

Show working or sketch a graph to justify your answer.

Question 23 (6 Marks)

The price, $P(t)$, of a pineapple during an average year in Queensland can be modelled by the function $P(t) = 60 + 48\sin\left(\frac{2\pi t}{183}\right)$, where $P(t)$ is the price of a pineapple in cents and t is the number of days after 1st January 2020, for $0 \le t \le 366$.

(a) What is the minimum price of a pineapple during the year? **1**

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(b) Sketch the function $P(t)$ for $0 \le t \le 366$. **2**

(c) When will the price of a pineapple be 84 cents for the <u>second</u> time in 2020? **3**

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Question 24 (4 Marks)

A team of horticulturalists counted the number of plants in a field at the end of each month over a year, where p is the number of plants and n is the amount of rainfall in mm. This data is shown in the table below.

Average monthly rainfall n	5	7	28	11	15	18	8	22	9	25	20	10
Number of plants p	300	**X**	500	450	220	350	420	200	**X**	520	320	210

(a) Before they lost certain data values, they calculated Pearson's correlation coefficient to be 0.446, correct to 3 decimal places. Comment on the strength of the correlation of these data values. **1**

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(b) A certain horticulturalist remembered that the least squares regression line has equation $p = 7.065n + b$, where: **2**

- $m = 7.065$
- b is the vertical intercept
- $\overline{p}$ is the mean number of plants
- $\overline{n}$ is the mean amount of rainfall
- $\overline{p} = 31\,\overline{n}$

Calculate the value of b, correct to 2 decimal places, given $b = \overline{p} - m\overline{n}$.

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(c) Another horticulturalist explained that the lost values in the above table could be found by interpolation using the least squares regression line. Explain why the horticulturalist is most likely incorrect. **1**

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Question 25 (3 Marks)

The volume of liquid in a tank is given by the formula $V = \cos 2t - \sqrt{3}t + 1$, **3**

where V is the volume of liquid in litres and t is the time in seconds.

The liquid is leaking from the tank. At what time is the rate of change of the volume zero, given $0 < t < \frac{3\pi}{4}$? Express your answer in exact form.

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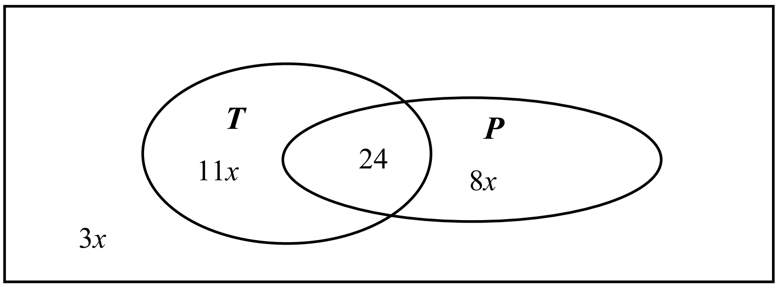
Question 26 (3 Marks)

The Venn diagram below shows the number of students who watch television (T) and listen to podcasts (P).



(a) If 244 students took part in the survey, show that the value of x is 10. **1**

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(b) What is the probability that a student chosen at random watches television? **2**

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Question 27 (7 Marks)

Let $f(x) = x^2 \ln x$, $x > 0$.

(a) Show that $f'(x) = x(2\ln x + 1)$. **2**

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(b) Find the stationary point of $f(x)$ and determine its nature. **2**

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(c) Show that there is a point of inflexion at approximately $(e^{\frac{-3}{2}}, -0.075)$. **3**

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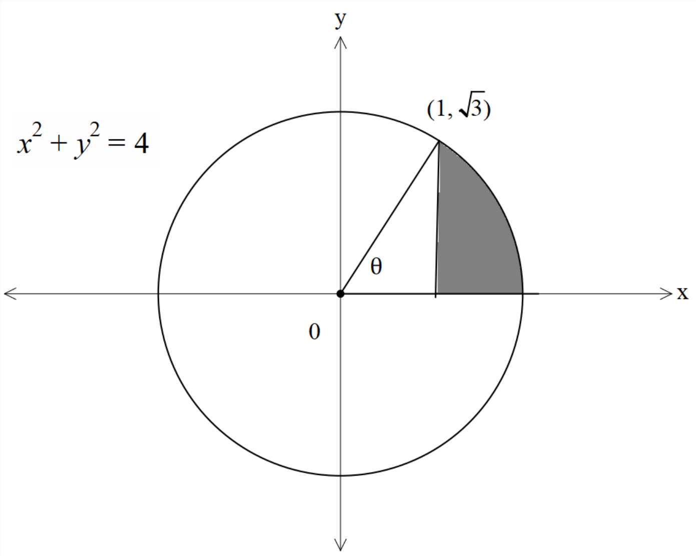
Question 28 (7 Marks)

The graph of the circle $x^2 + y^2 = 4$ is shown.

The interval connecting the origin, O, and the point $(1, \sqrt{3})$ makes an angle of θ with the positive direction of the x-axis .



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(a) By considering the value of θ, find the exact area of the shaded region, as shown in the diagram. **2**

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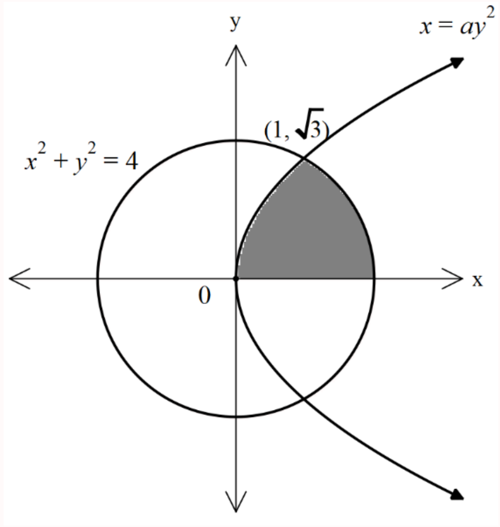
Question 28 continued

The parabola $x = ay^2$ which passes through the points $(0,0)$ and $(1,\sqrt{3})$ is drawn with the circle $x^2 + y^2 = 4$ as shown.



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(b) Find the value of a and show that the upper half of the parabola has equation $y = \sqrt{3x}$. **2**

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(c) Using parts (a) and (b), find the exact area of the region bounded by the upper half of the parabola, the positive direction of the x-axis and the circle as shown in the diagram. **3**

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Question 29 (4 Marks)

(a) Sketch $f(x)=\dfrac{1-x}{x-2}$. Clearly label asymptotes, x and y intercepts. **2**

(b) Find the equation of the tangent to $f(x)$ where $x=1$. **2**

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Question 30 (3 Marks)

Find the value of the constant p, where $p > 0$ such that **3**

$$\int_{\ln p}^{\ln p^2} \left(1 + \frac{1}{x}\right) dx = \ln 14 .$$

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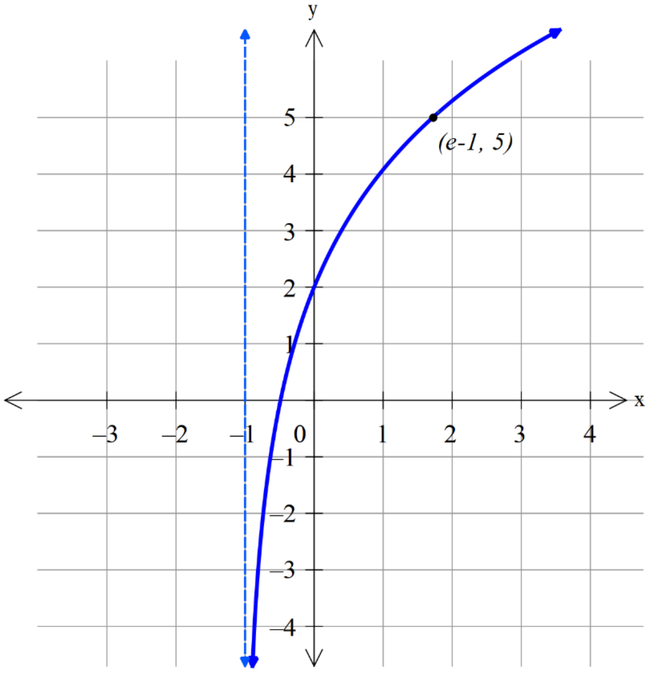
Question 31 (5 Marks)

The diagram below shows the graph of the function $f(x)=k\ln(x+a)+c$,

where $\boldsymbol{k}$, $\boldsymbol{a}$ and $\boldsymbol{c}$ are real constants.

- The graph has a vertical asymptote with equation $x=-1$.
- The graph has a y intersect at 2.
- The curve passes point $(e-1,5)$.

NOT TO SCALE



(a) Find the values of $\boldsymbol{k}$, $\boldsymbol{a}$ and $\boldsymbol{c}$, hence show that $f(x)=3\ln(x+1)+2$. **3**

(b) Find the area bounded by the curve $f(x)$, the ***y***-axis and the line $y = 5$. **2**

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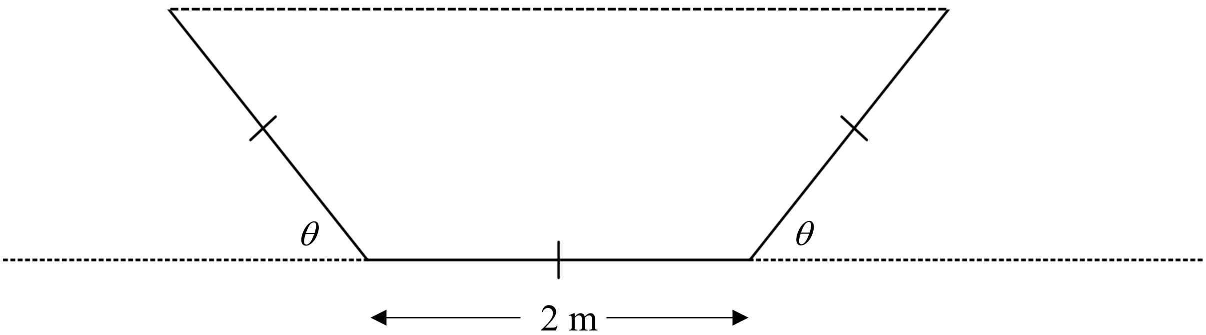
Question 32 (6 Marks)

A flat sheet of metal is bent at an angle of θ from the horizontal to form a cross-section of a drain, where $0 < \theta < \pi$. The resulting shape is an isosceles trapezium with three side lengths of 2 metres, as shown in the diagram below.

Not to scale



(a) Show that the cross-sectional area is given by $A = 4\sin\theta(1+\cos\theta)$. **2**

(b) Find the value of θ that maximises the cross-sectional area, and hence find the exact maximum area. **4**

Continue to Question 33 on next page

Question 33 (4 Marks)

A particle is undergoing straight line motion. At time t seconds it has displacement x metres from a fixed-point O on the line. The particle has velocity given by $v = 6e^t - e^{2t}$. Initially the particle is 5.5 metres to the right of O.

(a) Find when the particle changes direction, expressing your answer as an exact value. **1**

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(b) Hence, find the total distance travelled by the particle in the first $\log_e 12$ seconds. **3**

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End of Paper

HORNSBY GIRLS HIGH SCHOOL

Mathematics Advanced

Year 12 Higher School Certificate
Trial Examination Term 3 2023

STUDENT NUMBER: ______**SOLUTION AND MARKERS' FEEDBACK**________

General Instructions:

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
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Total Marks: 100

Section I – 10 marks (pages 3–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–30)

- Attempt Questions 11–33
- Allow about 2 hour and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. Let $f(x)=\frac{1}{\sqrt{x}}, g(x)=1-x$, the range of $g(f(x))$ is:

   A. $x>0$

   B. $y\neq 0$

   C. $y\in\mathbb{R}$

   D. $y<1$

$g(f(x))=1-\frac{1}{\sqrt{x}}$

$\frac{1}{\sqrt{x}}>0, x>0,$

$x\to+\infty, 1-\frac{1}{\sqrt{x}}\to 1$

$x\to 0, 1-\frac{1}{\sqrt{x}}\to -\infty$

2. The following table represents a probability distribution with a mean of 1.2.

x	*0*	*1*	*2*
$P(X=x)$	a	b	*0.3*

The value of a and b are:

   A. $a=0.1,\quad b=0.6$

   B. $a=0.6,\quad b=0.1$

   C. $a=0.3,\quad b=0.4$

   D. $a=0.5,\quad b=0.7$

$a+b+0.3=1$

$a+b=0.7$

$a(0)+b(1)+0.3(2)=1.2$

$b=1.2-0.6$

$=0.6$

$a=0.7-0.6$

$=0.1$

3. It is known that for a particular function, $y=f(x)$, that

$$f(2)=-5$$
$$f'(2)=0$$
$$f''(2)=3$$

Which statement below is true regarding the graph of $y=f(x)$?

   A. It passes through the point $(2,0)$.

   B. There is a local minimum at $(2,-5)$.

   C. It's concave down at $(2,-5)$.

   D. There is a point of inflexion at $(2,4)$.

At $x=2$, it's concave up, and it's a stationary point, therefore, a local minimum is at $(2,-5)$.

4. Which one of the following is equivalent to $\cot\theta$?

A. $\tan(2\pi-\theta)$

B. $\dfrac{1}{\tan\left(\dfrac{\pi}{2}-\theta\right)}$

C. $\sec^2\theta-1$

D. $\sin\theta\cos\theta\text{cosec}^2\theta$

$$\tan(2\pi-\theta)=-\tan\theta$$

$$\frac{1}{\tan\left(\frac{\pi}{2}-\theta\right)}=\frac{1}{\cot\theta}=\tan\theta$$

$$\sec^2\theta-1=\tan^2\theta$$

$$\sin\theta\cos\theta\cos ec^2\theta=\sin\theta\cos\theta\frac{1}{\sin^2\theta}$$

$$=\frac{\cos\theta}{\sin\theta}$$

$$=\cot\theta$$

5. The graph of the function $y=f(x)$ is known to have a minimum turning pointing at the point $P(-4,-8)$. Therefore, the graph of $y=-f(2x)$ will have a maximum turning point at:

A. $(-2,8)$

B. $(-8,8)$

C. $(-4,8)$

D. $(2,-8)$

The y value is flipped on x-axis and becomes 8. The x value is halved and becomes -2.

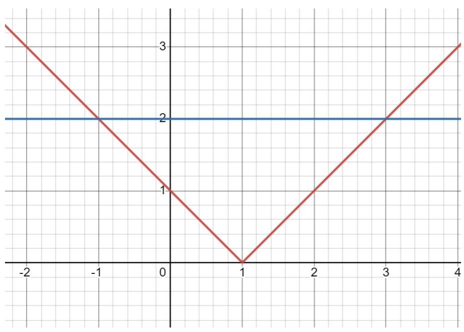
6. The solution of $|x-1|>2$ is:

A. $x>3$

B. $x<-1, x>3$

C. $x<-1$

D. $x\le -1, x\ge 3$



7. Which one of the following statements is not true?

A. $\int_{-3}^{3} x^3 dx = 0$

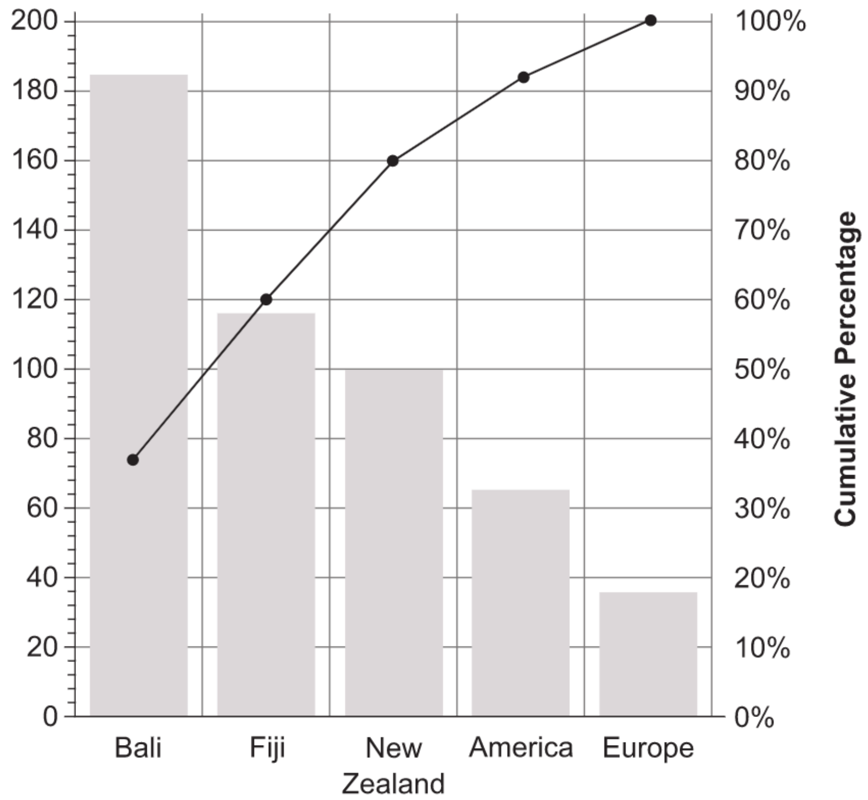
B. $\int_{-3}^{3} x^2 dx = 2\int_{-3}^{0} x^2 dx$

C. $\int_{0}^{3} x dx = -\int_{-3}^{0} x dx$

D. $\int_{-1}^{2} 3dx = 6$

$$\int_{-1}^{2} 3dx = 3\times 3 = 9$$

8. The Pareto chart below shows the data collected from a survey where people were asked to choose their favourite overseas holiday destination.



Using the chart, which one of the following is true?

A. 80% of the people chose Bali and Fiji as their favourite holiday destination.

B. 50% of the people chose New Zealand as their favourite holiday destination.

C. 100 people chose Fiji as their favourite holiday destination.

D. A total of 500 people were surveyed.

100 people which is 20% chose NZ. Total number of people $= 100 \div 0.2 = 500$

9. For which domain is the expression $\cos\theta + \cot\theta < 0$ true?

A. $\left(0, \frac{\pi}{2}\right)$

B. $\left(\pi, \frac{3\pi}{2}\right)$

C. $\left(\frac{\pi}{2}, \pi\right)$

D. $\left(-\pi, -\frac{\pi}{2}\right)$

$$\cos\theta + \frac{\cos\theta}{\sin\theta} < 0$$

$$\cos\theta\left(1 + \frac{1}{\sin\theta}\right) < 0$$

$Q1: \cos\theta > 0$ and $1 + \frac{1}{\sin\theta} > 0$

$Q2: \cos\theta < 0$ and $1 + \frac{1}{\sin\theta} > 0$

$Q3: \cos\theta < 0$ and $1 + \frac{1}{\sin\theta} < 0$

$Q4: \cos\theta > 0$ and $1 + \frac{1}{\sin\theta} < 0$

Therefore, the expression is true for angles in Q2 and Q4.

10. Which function is a primitive of $4x\left(x^2+3\right)^5$?

A. $12\left(x^2+1\right)\left(x^2+3\right)^4 + C$

B. $\frac{\left(x^2+3\right)^6}{3} + C$

C. $\frac{2\left(x^2+3\right)^6}{3} + C$

D. $\left(x^2+3\right)^6 + C$

$$\int 4x\left(x^2+3\right)^5 dx = 2\int 2x\left(x^2+3\right)^5 dx$$

$$= \frac{2}{6}\left(x^2+3\right)^6$$

$$= \frac{\left(x^2+3\right)^6}{3}$$

End of Section I

Section II 90 marks

Attempt all questions. Allow about 2 hours and 45 minutes for this section

Answer each question in the spaces provided.

Your responses should include relevant mathematical reasoning and/or calculations.

Extra writing space is provided at the back of the examination paper.

Question 11 (3 Marks)

20 people had their heart rate measured in beats per minute after a period of exercise.

The information is displayed below in a stem and leaf plot.

Heart Rate After Exercise

8	**6 6 7 8 9**
9	**0 3 4 4 5 8 8 9**
10	**1 3 3**
11	**7**
12	**4 5**
13	
14	**3**

(a) Use calculations to explain why the heart rate of 143 beats per minute after exercise could be considered an outlier for the recorded data. **2**

IQR = Q3 − Q1 $\quad$ Q1 = $\frac{89+90}{2}$ = 89.5 $\quad$ Q3 = 103

= 103 − 89.5

= 13.5

Q3 + 1.5 × IQR

= 103 + 1.5 × 13.5

= 123.25

∴ 143 b/m > 123.25 hence the score is considered to be an outlier.

Conclusion needs to be justified correctly. Many students used incorrect formula. Note the formula is on the reference sheet.

(b) Describe the skewness of the heart rate data after exercise. **1**

Positively skewed.

Many chose Negatively skewed.

Question 12 (4 Marks)

Metal spherical balls are manufactured so that their weight, W, varies directly with the cube of their radii, r. A ball manufactured with a radius of 2.5 cm, has a weight of 1.25 kg. **2**

(a) Calculate the radius of a ball manufactured with a weight of 5.12 kg.

$W \propto r^3$

$W = kr^3$

when $r = 2.5$ $w = 1.25$

$\therefore 1.25 = k(2.5)^3$

$k = \frac{1.25}{(2.5)^3}$

$= 0.08$

$\therefore W = 0.08r^3$

when $r = ?$ $w = 5.12$

$5.12 = 0.08 r^3$

$r^3 = \frac{5.12}{0.08}$

$r = \sqrt[3]{\frac{5.12}{0.08}}$

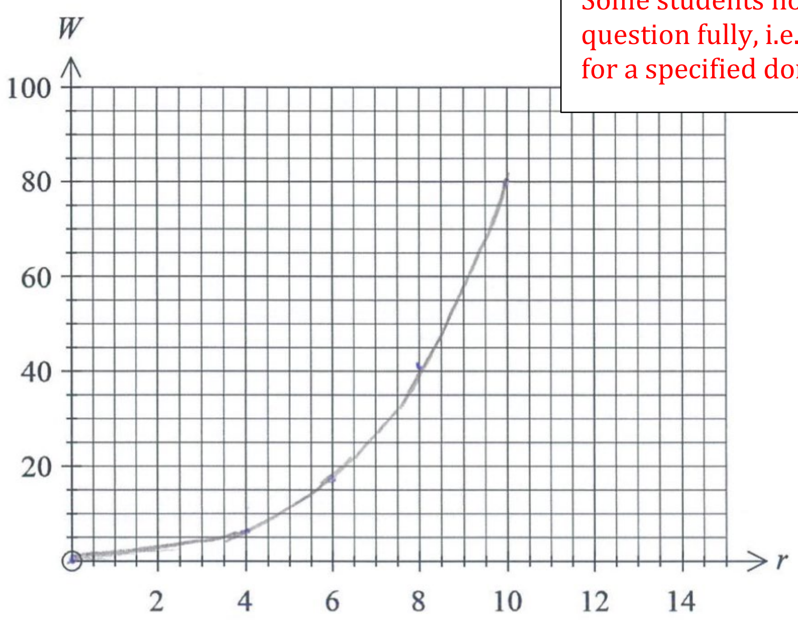
$= 4$

$\therefore$ A radius of 4 cm will produce a ball of weight 5.12 kg.

This question was surprising not done very well.
Many considered ratios and got the correct answer.
There were many who had a linear relationship, it was a cubic relationship!

(b) Graph the relationship between the radius and weight of the metal spherical balls from $r = 0$ cm to $r = 10$ cm. **2**

A variety of acceptable solutions based on carry on error from part (a).
Some students not reading the question fully, i.e., graph drawn for a specified domain.



Question 13 (2 Marks)

Use the trapezoidal rule with three subintervals to find an approximate value of

$$\int_1^7 \frac{1}{\sqrt{2x-1}}\,dx.$$

Give your answer correct to 2 decimal places.

Many students took 3 subintervals for 3 function values!
Need to take care to not round off to early, if not using exact values in the table.

$\int \frac{1}{\sqrt{2x-1}}\,dx$

x	1	3	5	7
$f(x)$	1	$\frac{1}{\sqrt{5}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{13}}$

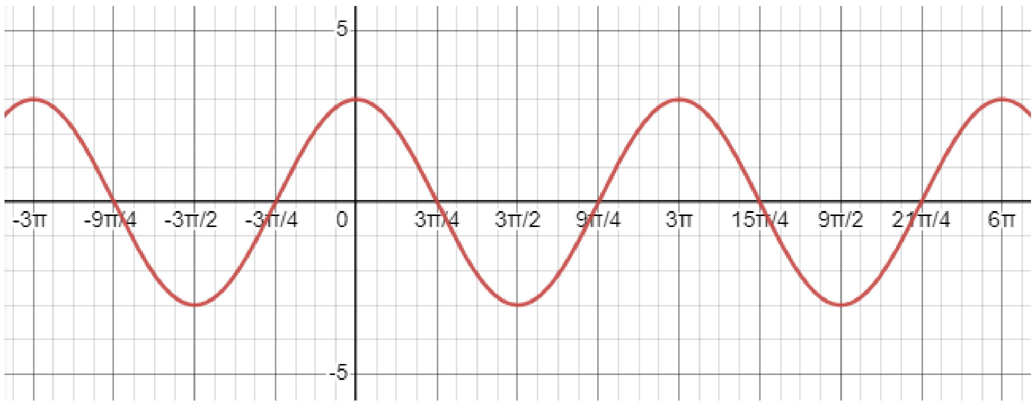
$= \frac{2}{2}\left[1 + \frac{1}{\sqrt{13}} + 2\left(\frac{1}{\sqrt{5}} + \frac{1}{3}\right)\right]$

$= 2.83844\ldots$

$\doteq 2.84$ (2dp)

Question 14 (2 Marks)

The graph of $y = k\cos nx$ is shown.



What are the values of k and n?

amplitude, $k = 3$

Done well

period $= \frac{2\pi}{n} = 3\pi$

$\frac{2\pi}{3\pi} = n$

$n = \frac{2}{3}$

Question 15 (3 Marks)

Three standard six-sided dice are thrown once.

a) What is the probability that all three dice show six? **1**

$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$

b) What is the probability that exactly two of the dice show six? **1**

$\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times 3$ (not 6 could be on die 1, 2 or 3)

$= \frac{3 \times 5}{216}$

$= \frac{5}{72}$

Some students did not multiply by 3

c) What is the probability of rolling three different numbers? **1**

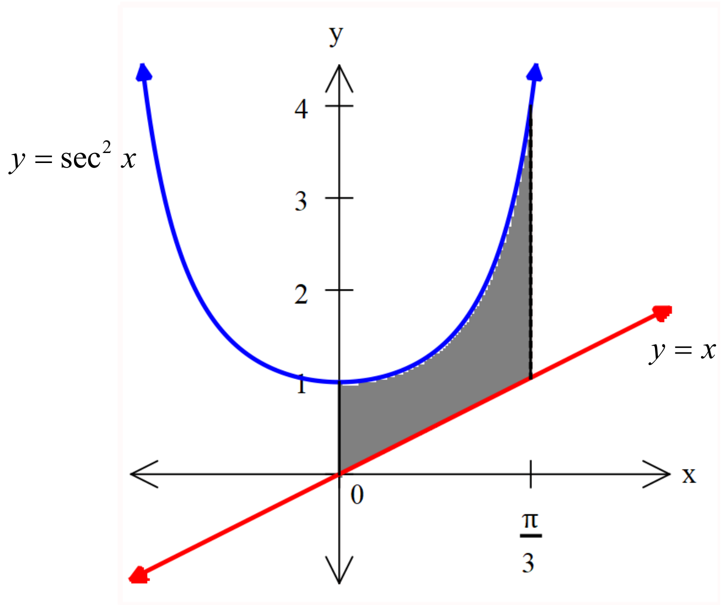
Die 1 Die 2 Die 3

$\frac{6}{6} \times \frac{5}{6} \times \frac{4}{6} = \frac{20}{36} = \frac{5}{9}$

Question 16 (3 Marks)

The diagram shows the graphs of the functions $y = \sec^2 x$ and $y = x$.

Calculate the area of the shaded region between $x = 0$ and $x = \frac{\pi}{3}$.



NOT TO SCALE

$A = \int_0^{\frac{\pi}{3}} (\sec^2 x - x)\,.dx$ — 1 for setting up correct integral

$= \left[\tan x - \frac{x^2}{2}\right]_0^{\frac{\pi}{3}}$ — 1 for correct integration

$= \tan\frac{\pi}{3} - \frac{\left(\frac{\pi}{3}\right)^2}{2} - \left(\tan 0 - \frac{0^2}{2}\right)$

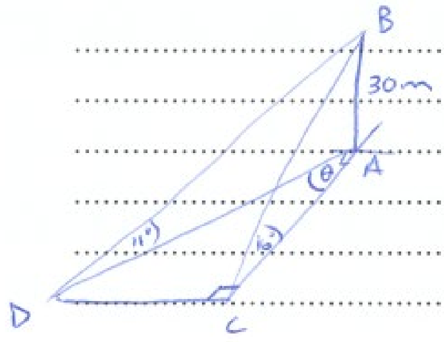
$= \left(\sqrt{3} - \frac{\pi^2}{18}\right)$ units^2 — 1 for correct substitution and evaluation

Many students made errors with $\frac{\left(\frac{\pi}{3}\right)^2}{2} \neq \frac{2\pi^2}{9}$

Some made an error with $\int \sec^2 x\,dx \neq \tan^2 x$

Question 17 (4 Marks)

AB is a vertical tower 30 m high, C and D are points at the same level as the foot of tower A. The point C is due south of the tower and D is due west of C. If the angles of elevation of C and D to the top of the tower are 16^0 and 11^0 respectively, find the distance CD and the bearing of D from the tower. Correct your answer to 2 decimal places and nearest minutes.



In $\triangle ABC$

$\tan 16° = \frac{30}{AC}$

$AC = \frac{30}{\tan 16°}$

$AC = 30 \cot 16°$

In $\triangle ABD$

$\tan 11° = \frac{30}{AD}$

$AD = \frac{30}{\tan 11°}$

$AD = 30 \cot 11°$

In $\triangle ACD$

$AD^2 = AC^2 + CD^2$

$30^2 \cot^2 11° = 30^2 \cot^2 16° + CD^2$ 1

$CD^2 = 30^2(\cot^2 11° - \cot^2 16°)$

$CD^2 = 12873.93887$

$CD = 113.4633812..$

$CD \doteqdot 113.46\text{ m}$ 1 for CD

$\cos \angle CAD = \frac{AC}{AD}$ let $\angle CAD = \theta°$

$\cos \theta° = \frac{30 \cot 16°}{30 \cot 11°}$

$\cos \theta° = 0.677846977$

$\theta° = 47.32143382$

$\theta° = 47°19'$ 1 for angle

Bearing of D from A is $180° + \theta°$

$= 180 + 47°19'$

$= 227°19'$

1 for bearing

nearest degree is fine

……………………………………………………………………………or S47 $S47^0 19'W$

Many students had an incorrect (and easier) diagram with the right angle for $\triangle ACD$ at $\angle DAC$ instead of $\angle ACD$.

Question 18 (3 Marks)

(a) Differentiate $y = x\tan x$ **1**

$\frac{d}{dx} x\tan x = \tan x \times 1 + x \times \sec^2 x$

1 for correct use of product rule

Part (a) was done well, part (b) was not

(b) Hence, or otherwise, find $\int x\sec^2 x.dx$. **2**

$\int (\tan x + x\sec^2 x).dx = x\tan x + c$

$\int \tan x.dx + \int x\sec^2 x\,dx = x\tan x + c$

1 for rearrangement

$\int x\sec^2 x\,dx = x\tan x - \int \tan x.dx + c$

$= x\tan x - \int \frac{\sin x}{\cos x}.dx + c$

$= x\tan x - -\int \frac{-\sin x}{\cos x}.dx + c$

1 for $\int \tan x.dx$

$= x\tan x + \ln|\cos x| + k$

Question 19 (3 Marks)

The curve $y = kx^2 + c$ is subject to the following transformations:

- Translate 2 units to the right.
- Dilated vertically by a factor of 4.
- Reflected in the $\boldsymbol{y}$-axis.

The final equation of the curve is $y = 8x^2 + 32x - 8$.

Find the values of $\boldsymbol{k}$ and $\boldsymbol{c}$.

$$y = k(x-2)^2 + c$$

$$\frac{y}{4} = k(x-2)^2 + c$$

$$\frac{y}{4} = k(-x-2)^2 + c$$

$$\frac{y}{4} = k(x+2)^2 + c$$

$$y = 4k(x^2 + 4x + 4) + 4c$$

$$y = 4kx^2 + 16kx + 16k + 4c$$

$$4k = 8$$

$$\therefore k = 2$$

$$16k + 4c = -8$$

$$32 + 4c = -8$$

$$\therefore c = -10$$

Generally, well done. Some students didn't read the question carefully. They need to transform $y = kx^2 + c$ into $y = 8x^2 + 32x - 8$. Not vice versa.

Question 20 (4 Marks)

The population of magpies, P, in a suburb is modelled by the function $P = \dfrac{300}{1+14e^{-0.5t}}$, where t is the time in years.

(a) Find the range of the function P. **2**

$$t=0, P=\frac{300}{1+14e^{-0.5(0)}}$$
$$=\frac{300}{1+14}$$
$$=20$$
$$t\ge 0, t\to +\infty, e^{-0.5t}\to 0$$
$$\frac{300}{1+14e^{-0.5t}}\to 300$$
$$\therefore 20\le P<300$$

Some students were not able to find the limit of the function. Note: $t \ge 0$, and $P \neq 300$.

(b) What will be the rate of change of the population of magpies after 5 years? Answer correct to the nearest whole number. **2**

$$P=300\left(1+14e^{-0.5t}\right)^{-1}$$
$$\frac{dP}{dt}=\frac{300(-1)14(-0.5)e^{-0.5t}}{\left(1+14e^{-0.5t}\right)^2}$$
$$=\frac{2100e^{-0.5t}}{\left(1+14e^{-0.5t}\right)^2}$$
$$t=5$$
$$\frac{dP}{dt}=\frac{2100e^{-0.5(5)}}{\left(1+14e^{-0.5(5)}\right)^2}$$
$$=37.31929883....$$
$$\approx 37 \text{ per year}$$

Most of the students did well. Some didn't differentiate the function correctly.

Question 21 (4 Marks)

(a) A university is made up of local and international students. 85% are local students of whom 25% hold a scholarship. Of the international students, 45% are scholarship holders. If a randomly chosen student holds a scholarship, what is the probability that he or she is an international student? **2**

Let I be the event *'the student is an international student.'*

Let S be the event *'the student holds a scholarship.'*

$$P(I|S) = \frac{P(I \cap S)}{P(S)}$$
$$= \frac{0.15 \times 0.45}{0.85 \times 0.25 + 0.15 \times 0.45}$$
$$= \frac{27}{112}$$

Most of the students did well.

(b) Suppose that A and B are not independent events. $P(A) = \frac{1}{4}, P(B) = \frac{2}{3}, P(A \cup B) = \frac{3}{4}$. **2**

If $\overline{A}$ denotes the complement of A, find $P\left(\overline{A} \cap B\right)$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\frac{3}{4} = \frac{1}{4} + \frac{2}{3} - P(A \cap B)$$
$$P(A \cap B) = \frac{1}{4} + \frac{2}{3} - \frac{3}{4}$$
$$= \frac{1}{6}$$
$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$
$$= \frac{2}{3} - \frac{1}{6}$$
$$= \frac{1}{2}$$

Many students were able to find $P(A \cap B)$ **, but not able to find** $P(\overline{A} \cap B)$.

Note that if the events are not independent, $P(\overline{A} \cap B) \neq P(\overline{A}) \times P(B)$

Question 22 (3 Marks)

Find the global maximum and minimum value of

$$y=\begin{cases}-2, & \text{for } x<-3\\ x+1, & \text{for } -3\le x<1\\ -x^3-x^2+4, & \text{for } 1\le x\le 1.5\end{cases}$$

Show working or sketch a graph to justify your answer.

for $y=-x^3-x^2+4,$

$y'=-3x^2-2x$

$y'=0$ for stationary points

$-3x^2-2x=0$

$x(3x+2)=0$

$x=0,-\dfrac{2}{3}$

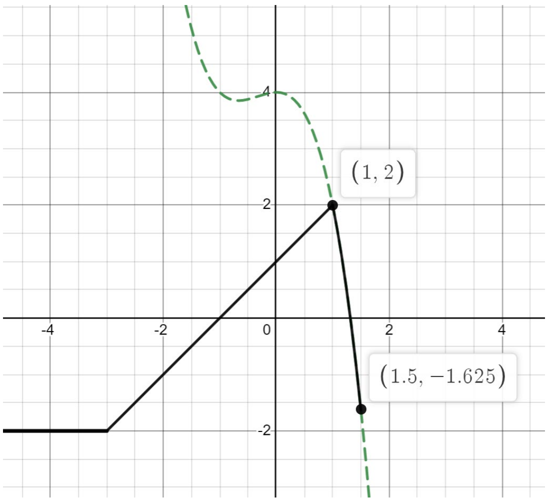
$y''=-6x-2$

$x=0, y''=-2<0,$ concave down

$\therefore$ local max at $x=0$

$x=-\dfrac{2}{3}, y''=-6\left(-\dfrac{2}{3}\right)-2=2>0,$ concave up

$\therefore$ local min at $x=-\dfrac{2}{3}$



However, the max and min values are not in the domain.
At $x=1.5, y=-1.625$. $x=1, y=2$.
Therefore, the global max value is 2 and the global min value is -2.

1 mark for max. value

1 mark for min. value

1 mark for correct graph or working to justify the answers

Many students managed to sketch the graph correctly to find the answers.

Note that the cubic function has a negative direction.

Some students answered the coordinates of the maximum and minimum points, which is not correct. The max/min values are the y-values.

Question 23 (6 Marks)

The price, $P(t)$, of a pineapple during an average year in Queensland can be modelled by the function $P(t) = 60 + 48\sin\left(\frac{2\pi t}{183}\right)$, where $P(t)$ is the price of a pineapple in cents and t is the number of days after 1st January 2020, for $0 \le t \le 366$.

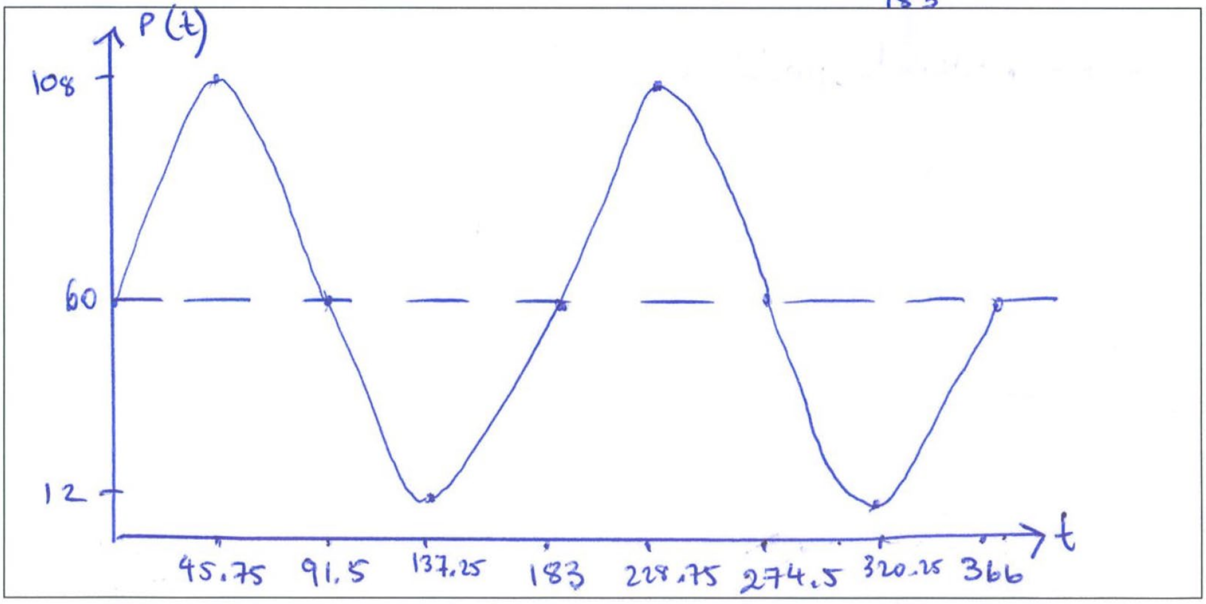
(a) What is the minimum price of a pineapple during the year? **1**

minimum price $= 60 - 48$

$= 12^c$

Generally well done

(b) Sketch the function $P(t)$ for $0 \le t \le 366$. $a = 48$, $p = \frac{2\pi}{\frac{2\pi}{183}} = 183$, centre $= 60$ **2**



Well done!

(c) When will the price of a pineapple be 84 cents for the second time in 2020? **3**

$84 = 60 + 48\sin\left(\frac{2\pi}{183}t\right)$

$\frac{24}{48} = \sin\left(\frac{2\pi}{183}t\right)$

$0.5 = \sin\left(\frac{2\pi}{183}t\right)$

$\frac{2\pi}{183}t = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore \frac{2\pi}{183}t = \frac{5\pi}{6}$ (second time)

$t = 76.25$ days.

Poorly done. Many students got the quadrants

Question 24 (4 Marks)

A team of horticulturalists counted the number of plants in a field at the end of each month over a year, where p is the number of plants and n is the amount of rainfall in mm.

This data is shown in the table below.

Average monthly rainfall n	5	7	28	11	15	18	8	22	9	25	20	10
Number of plants p	300	**X**	500	450	220	350	420	200	**X**	520	320	210

(a) Before they lost certain data values, they calculated Pearson's correlation coefficient to be 0.446, correct to 3 decimal places. Comment on the strength of the correlation of these data values. **1**

weak – moderate, positive

Very well done!

(b) A certain horticulturalist remembered that the least squares regression line has equation $p = 7.065n + b$, where: **2**

- $m = 7.065$
- b is the vertical intercept
- $\overline{p}$ is the mean number of plants
- $\overline{n}$ is the mean amount of rainfall
- $\overline{p} = 31\,\overline{n}$

Calculate the value of b, correct to 2 decimal places, given $b = \overline{p} - m\overline{n}$.

$\overline{n} = 14.8\dot{3}$ $\quad b = 459.8\dot{3} - 7.065 \times 14.8\dot{3}$

$\overline{p} = 31 \times 14.8\dot{3}$ $\quad b = 355.04$

$\overline{p} = 459.8\dot{3}$

Poorly done! Many students couldn't work out how everything was connected.

(c) Another horticulturalist explained that the lost values in the above table could be found by interpolation using the least squares regression line. Explain why the horticulturalist is incorrect. **1**

Since the correlation is weak – moderate, it is highly unlikely that the missing points will lie on the least squares regression line.

Generally well done

Question 25 (3 Marks)

The volume of liquid in a tank is given by the formula $V = \cos 2t - \sqrt{3}t + 1$, where V is the volume of liquid in Litres and t is the time in seconds. The liquid is leaking from the tank. At what time is the rate of change of the volume zero, given $0 < t < \frac{3\pi}{2}$?

Express your answer in exact form.

$V = \cos 2t - \sqrt{3}t + 1$ Poorly done……

$\frac{dV}{dt} = -2\sin 2t - \sqrt{3} = 0$

$-2\sin 2t = \sqrt{3}$

$\sin 2t = -\frac{\sqrt{3}}{2}$

angle $2t = \frac{\pi}{3}$

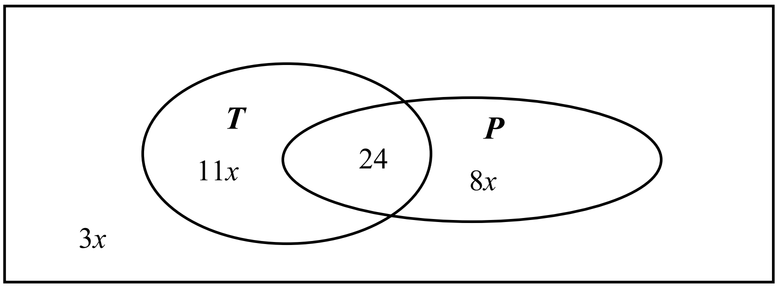
$2t = \frac{4\pi}{3}, \frac{5\pi}{3}$

$t = \frac{2\pi}{3}, \frac{5\pi}{6}$

(3 Marks) since $0 < t < \frac{3\pi}{4}$, $t = \frac{2\pi}{3}$

Question 26 (3 Marks)

The Venn diagram below shows the number of students who watch television (T) and listen to podcasts (P).



(a) If 244 students took part in the survey, what is the value of x? **1**

$11x + 8x + 24 + 3x = 244$ Very well done!

$22x = 220$

(b) $x = 10$ random watches television? **2**

Very well done!

$P(\text{watches television}) = \frac{24 + 110}{244}$

$= \frac{67}{122}$

Question 27 (7 Marks)

Let $f(x) = x^2 \ln x$.

(a) Show that $f'(x) = x(2\ln x + 1)$. **2**

$f'(x) = \ln x(2x) + x^2\left(\frac{1}{x}\right)$ Well done!

$= 2x(\ln x) + x$

$f'(x) = x(2\ln x + 1)$

(b) Find any stationary points of $f(x)$ and determine their nature. **3**

$0 = x(2\ln x + 1)$

$\therefore x = 0$ $\quad 2\ln x + 1 = 0$

$2\ln x = -1$

$\ln x = -\frac{1}{2}$

$x = e^{-\frac{1}{2}}$

when $x = 0$, $y = 0$ $\therefore (0,0)$

when $x = e^{-\frac{1}{2}}$, $y = \left(\frac{1}{\sqrt{e}}\right)^2\left(\ln e^{-\frac{1}{2}}\right)$

$y = \frac{1}{e} \times -\frac{1}{2}$

$y = \frac{-1}{2e}$

x	0.5	$e^{-\frac{1}{2}}$	1
y'	$-.2$	0	1

$\therefore$ local min. TP at $\left(\frac{1}{\sqrt{e}}, \frac{-1}{2e}\right)$

Well done.

(c) Show that there is a point of inflexion at approximately $(e^{-\frac{3}{2}}, -0.075)$. **2**

…Generally well done. Many students didn't show the full calculator reading for the y value.…

$f''(x) = x\left(\frac{2}{x}\right) + 2\ln x + 1$

$f''(x) = 2 + 2\ln x + 1$ $\quad$ Test for concavity

$f''(x) = 2\ln x + 3$

$0 = 2\ln x + 3$

$-\frac{3}{2} = \ln x$

$\therefore x = e^{-\frac{3}{2}} \approx 0.2231$

$y = \left(e^{-\frac{3}{2}}\right)^2\left(\ln e^{-\frac{3}{2}}\right)$

$y \doteqdot -0.07468\ldots$

$y \doteqdot -0.075$

x	0.1	$e^{-\frac{3}{2}}$	1
y''	-1.6	0	3

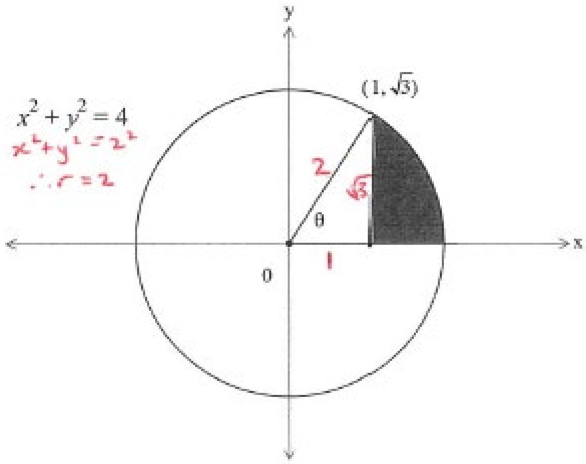
Change in concavity

$\therefore$ p.o.i at $\left(e^{-\frac{3}{2}}, -0.075\right)$

Question 28 (7 Marks)

The graph of the circle $x^2 + y^2 = 4$ is shown.

The interval connecting the origin, O, and the point $(1, \sqrt{3})$ makes an angle of θ with the positive direction of the x-axis .



NOT TO SCALE

(a) By considering the value of θ, find the exact area of the shaded region, as shown in the diagram. **2**

Done well by most students

$\tan\theta = \frac{\sqrt{3}}{1}$

$\theta = \frac{\pi}{3}$

Area of sector $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times 2^2 \times \frac{\pi}{3}$

$= \frac{2\pi}{3}$ (1)

$r = 2$ not $\sqrt{3}$ gives $\frac{\pi}{2} - \frac{\sqrt{3}}{2}$

Area of shaded region = area of sector − area of triangle

$= \frac{2\pi}{3} - \frac{1}{2} \times 1 \times \sqrt{3}$

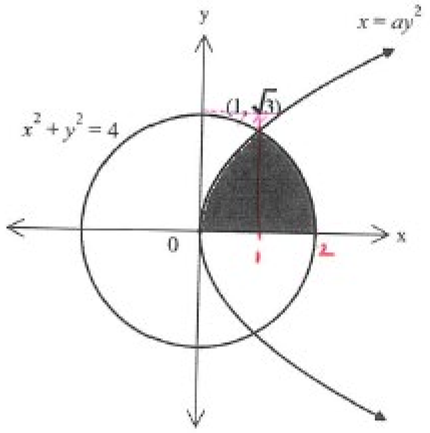
(or $\frac{1}{2} \times 1 \times 2 \times \sin\frac{\pi}{3}$)

$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ (1)

$= \frac{4\pi - 3\sqrt{3}}{6}$

Question 28 continued

The parabola $x = ay^2$ which passes through the points $(0,0)$ and $(1, \sqrt{3})$ is drawn with the circle $x^2 + y^2 = 4$ as shown.



Incorrect method
Some students tried
$\int_0^2 x \cdot dy$
$= \int_0^2 \frac{1}{3} y^2 \cdot dy$
$= \frac{1}{3} \int_0^2 y^2 \cdot dy$
$= \frac{1}{3} \left[\frac{y^3}{3}\right]_0^2$
$= \frac{1}{3} \times \frac{1}{3} (8 - 0)$
$= \frac{8}{9}$
and then did
$A_{quadrant} - \int_0^2 x \cdot dy$
but that does not consider the small area above the required region.

(b) Find the value of a and show that the upper half of the parabola has equation $y = \sqrt{3x}$. **2**

$x = ay^2$ passes through $(1, \sqrt{3})$
$1 = a \times (\sqrt{3})^2$
$1 = 3a$
$a = \frac{1}{3}$ (1)

$\therefore x = \frac{1}{3} y^2$
$3x = y^2$
$y = \pm\sqrt{3x}$
but $y \geq 0$ for the upper half
$\therefore y = \sqrt{3x}$

This working is needed for a "show that" question

(c) Using parts (a) and (b), find the exact area of the region bounded by the upper half of the parabola, the positive direction of the x-axis and the circle as shown in the diagram. **3**

Students who tried $\int_1^2 (4-x^2)^{\frac{1}{2}} \cdot dx$ were not successful

$A = \int_0^1 (3x)^{\frac{1}{2}} dx + \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ — 1 for correct integral statement

$= \left[\frac{(3x)^{3/2}}{\frac{3}{2} \times 3}\right]_0^1 + \frac{4\pi - 3\sqrt{3}}{6}$ — 1 for correct integration; 1 for correct simplification

$= \frac{2}{9}\left[3^{3/2} - 0^{3/2}\right] + \frac{4\pi - 3\sqrt{3}}{6}$

$= \frac{2}{9} \times \sqrt{27} + \frac{4\pi - 3\sqrt{3}}{6}$

$= \frac{2}{9} \times 3\sqrt{3} + \frac{4\pi - 3\sqrt{3}}{6}$

$= \frac{2\sqrt{3}}{3} + \frac{4\pi - 3\sqrt{3}}{6}$

$= \frac{4\sqrt{3}}{6} + \frac{4\pi - 3\sqrt{3}}{6}$

$= \frac{\sqrt{3} + 4\pi}{6}$

Many students forgot to divide by the derivative of the function!

OR
$\int_0^1 3^{\frac{1}{2}} \times x^{\frac{1}{2}} \cdot dx$
$= \sqrt{3} \int_0^1 x^{\frac{1}{2}} \cdot dx$
$= \sqrt{3} \times \left[\frac{x^{3/2}}{3/2}\right]_0^1$
$= \frac{2}{3} \times \sqrt{3} \, [1 - 0]$
$= \frac{2\sqrt{3}}{3}$

Students who used this method were more successful

23

Question 29 (4 Marks)

Consider the function $f(x)=\dfrac{1-x}{x-2}$.

(a) Sketch $f(x)=\dfrac{1-x}{x-2}$. Clearly label asymptotes, x and y intercepts. **2**

$f(x)=\dfrac{1-x}{x-2}$

$=\dfrac{-(x-2)-1}{x-2}$

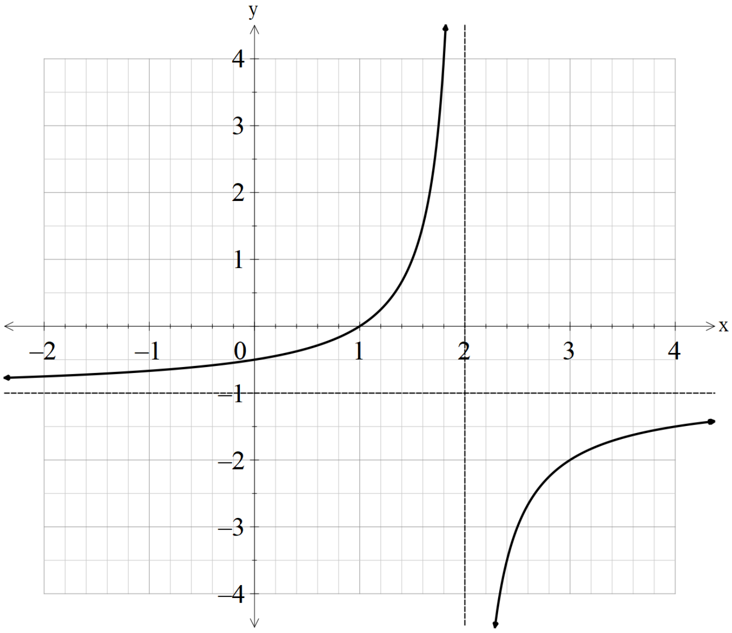
$=-1-\dfrac{1}{x-2}$

$\therefore$ the vertical asymptote is $x=2$

the horizontal asymptote is $y=-1$

x-intercept is $(1,0)$

y-intercept is $\left(0,\dfrac{-1}{2}\right)$



Most of the students did well. Some students didn't find the correct horizontal asymptotes.

(b) Find the equation of the tangent to the curve where $x=1$. **2**

$f'(x)=\dfrac{-1(x-2)-(1-x)1}{(x-2)^2}$

$=\dfrac{-x+2-1+x}{(x-2)^2}$

$=\dfrac{1}{(x-2)^2}$

$x=1, y=\dfrac{1-1}{1-2}=0, m=\dfrac{1}{(1-2)^2}=1$

$\therefore$ the equation of the tangent is $y=x-1$

Mostly well done. Some students didn't differentiate the function correctly using quotient rule.

Question 30 (3 Marks)

Find the value of the constant p, where $p > 0$ such that $\int_{\ln p}^{\ln p^2} \left(1+\frac{1}{x}\right) dx = \ln 14$.

$\left[x + \log_e |x|\right]_{\ln p}^{\ln p^2} = \ln 14$

$\left[\log_e p^2 + \log_e (\log_e p^2)\right] - \left[\log_e p + \log_e (\log_e p)\right] = \log_e 14$.

$\left[2\log_e p + \log_e (2\log_e p)\right] - \left[\log_e p + \log_e (\log_e p)\right] = \log_e 14$.

$2\log_e p + \log_e 2 + \log_e (\log_e p) - \log_e p - \log_e (\log_e p) = \log_e 14$.

$\log_e p + \log_e 2 = \log_e 14$

$\log_e p = \log_e 14 - \log_e 2$

$= \log_e \left(\frac{14}{2}\right)$

$= \log_e 7$

Equating: $\therefore p = 7$.

OR $\log_e p + \log_e 2 = \log_e 14$

$\log_e (2p) = \log_e 14$

Equating: $\therefore 2p = 14$

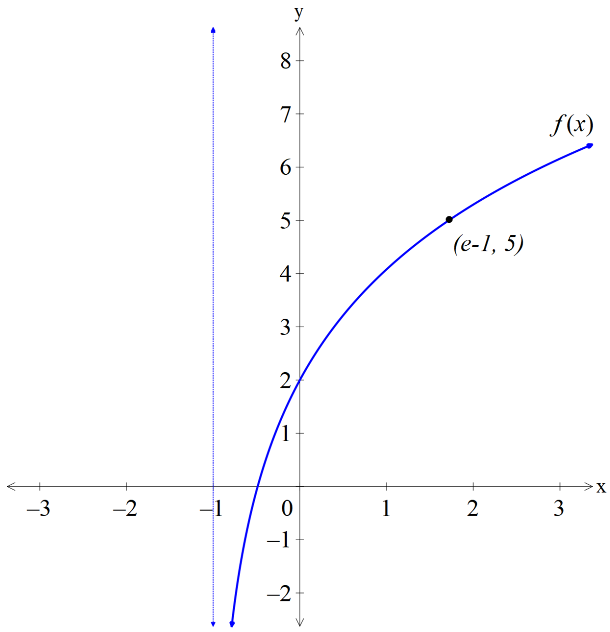
$p = 7$.

Many students did well in this question, there were different correct approaches.
Unfortunately some made a none attempt.

Question 31 (5 Marks)

The diagram below shows the graph of the function $f(x) = k\ln(x+a)+c$, where ***k***, ***a*** and ***c*** are real constants.

- The graph has a vertical asymptote with equation $x = -1$.
- The graph has a y intersect at 2.
- The curve passes point $(e-1, 5)$.



(a) Find the values of ***k***, ***a*** and ***c***, hence show that $f(x) = 3\ln(x+1)+2$. **3**

$x + a \neq 0$

$x \neq -a$

from the graph, $x \neq -1$

$\therefore a = 1, f(x) = k\ln(x+1)+c$

sub $x = 0, y = 2$

$k\ln(0+1)+c = 2$

$k(0)+c = 2$

$\therefore c = 2, f(x) = k\ln(x+1)+2$

sub $x = e-1, y = 5,$

$k\ln(e-1+1)+2 = 5$

$k\ln e + 2 = 5$

$k = 3,$

$\therefore f(x) = 3\ln(x+1)+2$

Students should show sufficient workings to justify their answers.

It is not sufficient to say c=2 because the y-intercept is 2. It is not always the case for log functions.

(b) Find the area bounded by the curve $f(x)$, the ***y***-axis and the line $y = 5$. **2**

$$y = 3\ln(x+1) + 2$$

$$\frac{y-2}{3} = \ln(x+1)$$

$$e^{\frac{y-2}{3}} = x+1$$

$$x = e^{\frac{y-2}{3}} - 1$$

$$\text{area} = \int_2^5 \left(e^{\frac{y-2}{3}} - 1 \right) dy$$

$$= \left[3e^{\frac{y-2}{3}} - y \right]_2^5$$

$$= 3e^{\frac{5-2}{3}} - 5 - \left(3e^{\frac{2-2}{3}} - 2 \right)$$

$$= 3e - 5 - 3 + 2$$

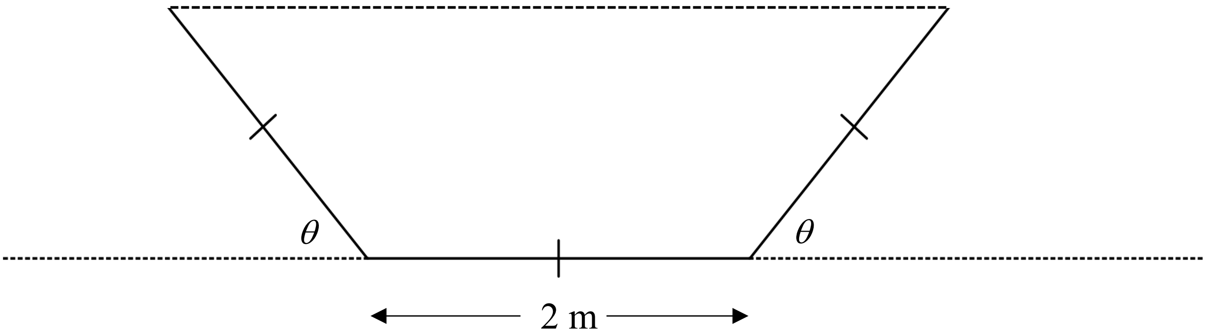
$$= 3e - 6$$

Many students did manage to convert the function into the subject of x, however, some didn't integrate the function correctly.

Question 32 (6 Marks)

A flat sheet of metal is bent at an angle of θ, from the horizontal to form a cross-section of a drain, where $0 < \theta < \pi$. The resulting shape is an isosceles trapezium with three side lengths of 2 metres. As shown in the diagram below.

Not to scale



(a) Show that the cross-sectional area is given by $A = 4\sin\theta(1+\cos\theta)$. **2**

$\sin\theta = \frac{h}{2}$ Let perpendicular height of trapezium be h.

$\therefore h = 2\sin\theta$

$\cos\theta = \frac{x}{2}$ Other side of trapezium be $2x+2$.

$x = 2\cos\theta$

Area of trapezium $= \frac{1}{2} \times h \times (a+b)$

$= \frac{1}{2} \times 2\sin\theta \times (2 + 2x + 2)$

$= \frac{1}{2} \times 2\sin\theta \times (4 + 2(2\cos\theta))$

$= \frac{1}{2} \times 2\sin\theta (4 + 4\cos\theta)$

$= 4(\sin\theta)(1 + \cos\theta)$ As required.

A few different approaches, done reasonably well by most students.
Students should introduce variables used.

(b) Find the value of θ that maximises the cross-sectional area, and hence find the exact maximum area. **4**

$A = 4\sin\theta(1+\cos\theta)$

$\frac{dA}{d\theta} = (1+\cos\theta).4\cos\theta + 4\sin\theta(-\sin\theta)$ $u = 4\sin\theta$ $v = 1+\cos\theta$

$u' = 4\cos\theta$ $v' = -\sin\theta$.

$= 4(\cos\theta + \cos^2\theta - \sin^2\theta)$

$= 4(\cos\theta + \cos^2\theta - (1 - \cos^2\theta))$

$= 4(\cos\theta + 2\cos^2\theta - 1)$

$= 4(2\cos^2\theta + \cos\theta - 1)$

Most students were able to give the correct derivative.

Now $\frac{dA}{d\theta} = 4(2\cos^2\theta + \cos\theta + 1)$

$= 4(2\cos\theta - 1)(\cos\theta + 1)$

For a maximum θ let $\frac{dA}{d\theta} = 0$

$\therefore 4(2\cos\theta - 1)(\cos\theta + 1) = 0$

$2\cos\theta - 1 = 0$ or $\cos\theta + 1 = 0$.

$2\cos\theta = 1$ $\quad$ $\cos\theta = -1$

$\cos\theta = \frac{1}{2}$ $\quad$ $\theta = \pi$

But $0 < \theta < \pi$

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$ $\quad$ $\therefore$ not a solution.

$= \frac{\pi}{3}$

This part was completed reasonably well by most students. However, many not considering the given domain in the question, this error was ignored.

Determine the nature at $\theta = \frac{\pi}{3}$

θ	$\pi/4$	$\pi/3$	$\pi/2$
$\frac{dA}{d\theta}$	2.82	0	−4
	/	—	\

$\frac{dA}{d\theta} = 4\left(\cos\frac{\pi}{4} + 1\right)\left(2\cos\frac{\pi}{4} - 1\right)$

$= 2.82$

$\frac{dA}{d\theta} = 4\left(\cos\frac{\pi}{2} + 1\right)\left(2\cos\frac{\pi}{2} - 1\right)$

$= -4$.

Hence a maximum Area when $\theta = \frac{\pi}{3}$

Area when $\theta = \frac{\pi}{3}$

$A = 4\sin\frac{\pi}{3}\left(1 + \cos\frac{\pi}{3}\right)$

$= 4 \times \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right)$

$= 4 \times \frac{\sqrt{3}}{2} \times \frac{3}{2}$

$= 3\sqrt{3}\ m^2$

A maximum area of $3\sqrt{3}m^2$ occurs when $\theta = \frac{\pi}{3}$.

Many are still not showing that a max occurs for $\theta = \frac{\pi}{3}$.

Also need to have values in the show part. Not sufficient to just have signs.

Quite a few did not read the hence part of the question!

Question 33 (4 Marks)

A particle is undergoing straight line motion. At time t seconds it has displacement x metres from a fixed-point O on the line. The particle has velocity given by $v = 6e^{t} - e^{2t}$. Initially the particle is 5.5 metres to the right of O.

(a) Find when the particle changes direction. Give answer as an exact value. **1**

Let $v = 0$ $\quad \therefore 6e^{t} - e^{2t} = 0$

$e^{t}(6 - e^{t}) = 0$

$e^{t} = 0$ or $6 - e^{t} = 0$

But $e^{t} > 0$ $\quad e^{t} = 6$

$\therefore t = \ln 6$.

Some found the displacement function in part (a).
This mark was given as being done for part (b).

(b) Find the total distance travelled by the particle in the first $\log_e 12$ seconds. **3**

$x = \int (6e^{t} - e^{2t})\,dt$.

$= 6e^{t} - \frac{e^{2t}}{2} + c$

When $t = 0$, $x = 5.5$

$\therefore 5.5 = 6e^{0} - \frac{e^{2(0)}}{2} + c$

$5.5 = 6 - \frac{1}{2} + c$

$\therefore c = 0$

$\therefore x = 6e^{t} - \frac{e^{2t}}{2}$

Those students who attempted the question made reasonable attempts; errors unfortunately made in calculations.

Space!

When $t = \ln 6$, $x = 6e^{\ln 6} - \frac{e^{2\ln 6}}{2}$

$= 6 \times 6 - \frac{6^2}{2}$

$= 18$

When $t = \ln 12$ $\quad x = 6e^{\ln 12} - \frac{e^{2\ln 12}}{2}$

$= 6 \times 12 - \frac{12^2}{2}$

$= 72 - 72 = 0$ (i.e. Back to the origin).

$\therefore$ Total distance travelled

$= (18 - 5.5) + 18 = 30.5$ metres

Alternative solution:

Total distance travelled= $\int_0^{\ln 6} \left(6e^t - e^{2t}\right)dt + \left|\int_{\ln 6}^{\ln 12} \left(6e^t - e^{2t}\right)dt\right|$

$= 12.5 + |-18|$

$= 30.5$

End of Paper